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Global variants of Hartogs' theorem

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Abstract. Hartogs' theorem asserts that a separately holomorphic function, defined on an open subset of $\mathbb{C}^n$, is holomorphic in all the variables. We prove a global variant of this theorem for functions defined on an open subset of the product of complex algebraic manifolds. We also obtain global Hartogs-type theorems for complex Nash functions and complex regular functions.

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1. Introduction. In this paper, by a *complex algebraic variety*, we mean a quasiprojective complex algebraic variety (not necessarily irreducible). A *complex algebraic manifold* is a nonsingular complex algebraic variety of pure dimension. We assume throughout that subvarieties are Zariski closed in the ambient variety. Unless explicitly stated otherwise, we always make use of the Euclidean (metric) topology on complex algebraic varieties. Due to the nature of the investigated problems, we employ terminology typical to analytic geometry and differential topology (submersion, regular value, etc.). All results stated in this section are proved in Section 2.

Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic varieties and let $\pi_i \colon X \to X_i$ be the canonical projection. We say that a subset A of X is *parallel to the i-th factor of X* if $\pi_j(A)$ consists of one point for each $j \neq i$.

One of the main goals of the present paper is the following.

Theorem 1.1. *Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic manifolds and let $f \colon U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$, parallel to one of the factors of X, the restriction $f|_{U\cap C}$ is a holomorphic function. Then f is a holomorphic function.*

Theorem 1.1 is a generalization of the classical theorem of Hartogs, which asserts that a separately holomorphic function $f\colon U \to \mathbb{C}$, defined on an open subset U of $\mathbb{C}^n$, is holomorphic in all the variables [1,5]. *Separately holomorphic* means precisely that for each affine line $L \subset \mathbb{C}^n$, parallel to one of the coordinate axes, the restriction $f|_{U\cap L}$ is a holomorphic function.

It is natural to expect that Theorem 1.1 can be deduced from Hartogs' theorem by means of local holomorphic charts on X. This approach indeed works, but it requires some additional insight. The local charts have to be chosen in a special way since only nonsingular Zariski closed curves $C \subset X$ are allowed in Theorem 1.1.

We also have a counterpart of Theorem 1.1 for complex Nash functions. For the sake of clarity, we first recall the relevant definition. Let X, Y be complex algebraic manifolds and let $U \subset X, V \subset Y$ be open subsets. A map $\varphi\colon U \to V$ is said to be a *Nash map* if it is holomorphic and each point $x \in U$ has an open neighborhood $U_x \subset U$ such that the graph of the restriction $\varphi|_{U_x}$ is contained in a complex algebraic subvariety of $X \times Y$ of dimension $\dim U_x (= \dim X)$ [2]. The composite of Nash maps is a Nash map. Nash isomorphisms are defined in the standard way.

Theorem 1.2. *Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic manifolds and let $f\colon U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$, parallel to one of the factors of X, the restriction $f|_{U\cap C}$ is a Nash function. Then f is a Nash function.*

Theorems 1.1 and 1.2 have a suitable analog for regular functions. Let X be a complex algebraic manifold. A function $f\colon U \to \mathbb{C}$, defined on an open subset U of X, is said to be *regular* if there exists a rational function R on X such that $U \subset X \setminus \mathrm{Pole}(R)$ and $f = R|_U$, where $\mathrm{Pole}(R)$ stands for the polar set of R. Clearly, any regular function on U is a Nash function.

Theorem 1.3. *Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic manifolds and let $f\colon U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$, parallel to one of the factors of X, the restriction $f|_{U\cap C}$ is a regular function. Then f is a regular function.*

In the proof of Theorem 1.3, one of the ingredients is [7, Theorem 7.3]. Both theorems are related but not quite directly; in the former we use nonsingular algebraic curves, while in the latter smooth arcs of (possibly singular) algebraic curves are used. Our Theorem 2.8, which is a slighty more technical variant of Theorem 1.3, is a sharpening of [7, Theorem 7.3].

Theorems 1.1, 1.2, and 1.3 can be viewed as global variants of Hartogs' theorem for the appropriate classes of functions. In each of them, the case $n = 1$ is crucial, and we single it out in Propositions 2.3, 2.4, and 2.6, respectively.

2. Proofs. We let $\mathbb{P}^n$ denote complex projective n-space, and identify $\mathbb{C}^n$ with a subset of $\mathbb{P}^n$ via the map

$$\mathbb{C}^n \to \mathbb{P}^n, \quad (z_1, \ldots, z_n) \mapsto (1 : z_1 : \cdots : z_n).$$

Thus $0 = (0, \dots, 0) \in \mathbb{C}^n \subset \mathbb{P}^n$.

The following version of the Noether normalization lemma will be useful (we were not able to find a reference for it).

Lemma 2.1. *Let X be a projective complex algebraic manifold of dimension n, and let p be a point in X. Then there exists a regular map $\varphi\colon X \to \mathbb{P}^n$ for which $0 \in \mathbb{P}^n$ is a regular value and $\varphi(p) = 0$.*

Proof. We may assume that X is an algebraic subvariety of $\mathbb{P}^m$ for some $m > n$, and the following hold:

- $p = (1 : 0 : \dots : 0) \in X$;
- $X \subset \mathbb{P}^m \setminus L$, where $L \subset \mathbb{P}^m$ is the projective subspace of dimension $m - n - 1$ defined by
$$L = \{(z_0 : \dots : z_m) \in \mathbb{P}^m\colon z_0 = 0, \dots, z_n = 0\};$$
- the regular map
$$\pi\colon X \to \mathbb{P}^n, (z_0 : \dots : z_m) \mapsto (z_0 : \dots : z_n)$$
is a submersion at the point p.

A map φ having the required properties can be constructed by perturbing π. To this end, let M be the space of all $(n+1)$-by-m matrices with complex entries. For any constant $\varepsilon > 0$, set

$$M_\varepsilon = \{t = (t_{ij}) \in M\colon |t_{ij}| < \varepsilon \text{ for } 0 \le i \le n, 1 \le j \le m\}.$$

The manifold X being compact, for any ε sufficiently small, we obtain a well-defined regular map $\Phi\colon X \times M_\varepsilon \to \mathbb{P}^n$,

$$\Phi(z,t) = \left(z_0 + \sum_{j=1}^{m} t_{0j} z_j \colon \ \cdots : z_n + \sum_{j=1}^{m} t_{nj} z_j \right),$$

where $z = (z_0 : \dots : z_m) \in X$ and $t = (t_{ij}) \in M_\varepsilon$.

If ε is small enough, the map Φ is a submersion since for each point $z \neq p$ the restriction of Φ to $\{z\} \times M_\varepsilon$ is a submersion, and π is a submersion at p. In particular, $0 \in \mathbb{P}^n$ is a regular value of Φ. Hence, according to the standard consequence of Sard's theorem [4, p. 79, Theorem 2.7], the point $0 \in \mathbb{P}^n$ is also a regular value of the map

$$\Phi_t\colon X \to \mathbb{P}^n, \quad \Phi_t(z) = \Phi(z,t)$$

for some $t \in M_\varepsilon$. The regular map $\varphi = \Phi_t$ satisfies all the requirements. $\square$

Let $\mathbb{G}_1(\mathbb{P}^n)$ (resp. $\mathbb{G}_1(\mathbb{C}^n)$) denote the Grassmann manifold of projective lines in $\mathbb{P}^n$ (resp. vector lines in $\mathbb{C}^n$, that is, one-dimensional linear subspaces), and set

$$\mathbb{G}_1(\mathbb{P}^n, 0) = \{L \in \mathbb{G}_1(\mathbb{P}^n)\colon 0 \in L\}.$$

It readily follows that $\mathbb{G}_1(\mathbb{P}^n, 0)$ is an algebraic submanifold of $\mathbb{G}_1(\mathbb{P}^n)$, biregularly isomorphic to $\mathbb{P}^{n-1} = \mathbb{G}_1(\mathbb{C}^n)$.

The map

$$\mathbb{G}_1(\mathbb{P}^n, 0) \to \mathbb{G}_1(\mathbb{C}^n), \quad L \mapsto L \cap \mathbb{C}^n$$

is a biregular isomorphism.

Our next auxiliary result is inspired by [6]. In its proof, we use basic notions and results from differential topology, all of which can be found in [4].

Lemma 2.2. *Let X be a projective complex algebraic manifold of dimension $n \geq 1$, and let $\varphi\colon X \to \mathbb{P}^n$ be a regular map for which $0 \in \mathbb{P}^n$ is a regular value. Then the set*

$$\Omega = \{L \in \mathbb{G}_1(\mathbb{P}^n)\colon \varphi \textit{ is transverse to } L\}$$

is open in $\mathbb{G}_1(\mathbb{P}^n)$, and the set

$$\Omega^0 = \Omega \cap \mathbb{G}_1(\mathbb{P}^n, 0)$$

is dense in $\mathbb{G}_1(\mathbb{P}^n, 0)$.

Proof. Consider the standard action of the general linear group $G = GL_{n+1}(\mathbb{C})$ on $\mathbb{P}^n$. Any element $\sigma \in G$ determines an automorphism

$$\widehat{\sigma}\colon \mathbb{P}^n \to \mathbb{P}^n.$$

The subgroup

$$G^0 = \{\sigma \in G\colon \widehat{\sigma}(0) = 0\}$$

of G acts on $\mathbb{P}^n \setminus \{0\}$, and this action is transitive. Moreover, the action of G on $\mathbb{G}_1(\mathbb{P}^n)$,

$$G \times \mathbb{G}_1(\mathbb{P}^n) \to \mathbb{G}_1(\mathbb{P}^n), (\sigma, L) \mapsto \widehat{\sigma}(L)$$

is transitive, and so is the induced action of G^0 on $\mathbb{G}_1(\mathbb{P}^n, 0)$.

Henceforth, we work with a fixed projective line $L_0 \in \mathbb{G}_1(\mathbb{P}^n, 0)$. The regular map

$$\alpha\colon G \to \mathbb{G}_1(\mathbb{P}^n), \sigma \mapsto \widehat{\sigma}(L_0)$$

is a submersion, and hence it is open. Moreover,

$$\Omega = \alpha(\Gamma) \quad \text{and} \quad \Omega^0 = \alpha(\Gamma^0),$$

where

$$\Gamma = \{\sigma \in G\colon \widehat{\sigma} \circ \varphi \text{ is transverse to } L_0\} \quad \text{and} \quad \Gamma^0 = \Gamma \cap G^0.$$

Thus, it suffices to prove that Γ is an open subset of G, and Γ^0 is a dense subset of G^0.

Step 1. The subset Γ is open in G.

Consider the space $\mathcal{C}^\infty(X, \mathbb{P}^n)$ of $\mathcal{C}^\infty$ maps, endowed with the $\mathcal{C}^\infty$ topology. The set

$$\mathcal{U} = \{f \in \mathcal{C}^\infty(X, \mathbb{P}^n)\colon f \text{ is transverse to } L_0\}$$

is open in $\mathcal{C}^\infty(X, \mathbb{P}^n)$, see [4, p. 74, Theorem 2.1]. Moreover, since the manifold X is compact, the map

$$\beta\colon G \to \mathcal{C}^\infty(X, \mathbb{P}^n), \quad \sigma \mapsto \widehat{\sigma} \circ \varphi$$

is continuous. The proof of Step 1 is complete since $\Gamma = \beta^{-1}(\mathcal{U})$.

Step 2. The subset Γ^0 is dense in G^0.

The regular map

$$\psi\colon G^0 \times X \to \mathbb{P}^n, \quad (\sigma, x) \mapsto \widehat{\sigma} \circ \varphi$$

is a submersion since $0 \in \mathbb{P}^n$ is a regular value of φ, and G^0 acts transitively on $\mathbb{P}^n \setminus \{0\}$. In particular, ψ is transverse to L_0. For any element $\sigma \in G^0$, let

$$\psi_\sigma\colon X \to \mathbb{P}^n$$

be the map defined by

$$\psi_\sigma(x) = \psi(\sigma, x) = (\widehat{\sigma} \circ \varphi)(x).$$

Clearly,

$$\Gamma^0 = \{\sigma \in G^0 \colon \psi_\sigma \text{ is transverse to } L_0\}.$$

By the standard transversality theorem [4, p. 79, Theorem 2.7], the set on the right hand side of the last equality is dense in G^0. The proof is complete. $\square$

The following will play a key role in the proof of Theorem 1.1.

Proposition 2.3. *Let X be a complex algebraic manifold and let $f\colon U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$ the restriction $f|_{U \cap C}$ is a holomorphic function. Then f is a holomorphic function.*

Proof. According to Hironaka's theorem on resolution of singularities [3], we may assume that the manifold X is projective. Moreover, it suffices to consider the case $n = \dim X > 1$. Now we are ready to apply Lemmas 2.1 and 2.2. Pick a point $p \in X$. By Lemma 2.1, there exists a regular map $\varphi\colon X \to \mathbb{P}^n$ for which $0 \in \mathbb{P}^n$ is a regular value and $\varphi(p) = 0$. In view of Lemma 2.2, the set

$$\Omega = \{L \in \mathbb{G}_1(\mathbb{P}^n) \colon \varphi \text{ is transverse to } L\}$$

is open in $\mathbb{G}_1(\mathbb{P}^n)$, and the set

$$\Omega^0 = \Omega \cap \mathbb{G}_1(\mathbb{P}^n, 0)$$

is dense in $\mathbb{G}_1(\mathbb{P}^n, 0)$. Hence, there exist n projective lines $L_1, \dots, L_n$ in Ω^0 such that the vector lines $L_1 \cap \mathbb{C}^n, \dots, L_n \cap \mathbb{C}^n$ in $\mathbb{C}^n$ are linearly independent. Changing coordinates in $\mathbb{C}^n$, we may assume that $L_i \cap \mathbb{C}^n$ is the i-th coordinate axis in $\mathbb{C}^n$ for $i = 1, \dots, n$. Given a constant $\varepsilon > 0$, we let $\mathcal{A}(\varepsilon)$ denote the set comprised of all affine lines $l \subset \mathbb{C}^n$ with the following properties:

- l is parallel to one of the coordinate axes;
- $l \cap P(\varepsilon) \neq \emptyset$, where $P(\varepsilon)$ is the polydisc

$$P(\varepsilon) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \colon |z_j| < \varepsilon \text{ for } j = 1, \dots, n\}.$$

The assignment $\lambda \mapsto L(\lambda)$, to every affine line $\lambda \subset \mathbb{C}^n$ its Zariski closure $L(\lambda) \subset \mathbb{P}^n$, gives an embedding of the Grassmann manifold of affine lines in $\mathbb{C}^n$ into the Grassmann manifold $\mathbb{G}_1(\mathbb{P}^n)$. Hence, since Ω is an open subset of $\mathbb{G}_1(\mathbb{P}^n)$, we can choose ε sufficiently small so that $L(l) \in \Omega$ for all $l \in \mathcal{A}(\varepsilon)$. It follows that for every $l \in \mathcal{A}(\varepsilon)$ the inverse image $C(l) = \varphi^{-1}(L(l))$ is a nonsingular algebraic curve in X.

We can complete the proof as follows. Choose an open neighborhood $U_p \subset U$ of p so that the set $W_p = \varphi(U_p)$ is contained in $P(\varepsilon)$ and the restriction

$$\varphi_p : U_p \to W_p$$

of φ is a biholomorphism. The composite function $f \circ \varphi_p^{-1} : W_p \to \mathbb{C}$ is separately holomorphic since for any affine line $l \in \mathcal{A}(\varepsilon)$ the restriction $f|_{U \cap C(l)}$ is a holomorphic function. Hence, by Hartogs' theorem, the function $f \circ \varphi_p^{-1}$ is holomorphic, which in turn implies holomorphicity of the restriction $f|_{U_p}$. Thus, f is a holomorphic function, the point $p \in U$ being arbitrary. □

Proof of Theorem 1.1. Pick a point $p = (p_1, \dots, p_n)$ in $X = X_1 \times \cdots \times X_n$. For $i = 1, \dots, n$, let

$$X(i) = Y_{i1} \times \cdots \times Y_{in},$$

where $Y_{ii} = X_i$ and $Y_{ij} = \{p_j\}$ if $i \neq j$. By Proposition 2.3, the restriction $f|_{U \cap X(i)}$ is a holomorphic function. It follows immediately that the function f is holomorphic in a neighborhood of the point p. Indeed, it suffices to choose a local holomorphic chart in a neighborhood of each point p_i, and then apply Hartogs' theorem. The proof is complete since p is an arbitrary point of X. □

We already recalled the notion of a Nash map (hence, in particular, Nash function) in the introduction. Clearly, a function $f : U \to \mathbb{C}$, defined on an open subset U of $\mathbb{C}^n$, is a Nash function if and only if it is holomorphic and algebraic. Here, *algebraic* means that for each connected component W of U there exists a nonzero polynomial function P on $\mathbb{C}^n \times \mathbb{C}$ with

$$P(z, f(z)) = 0 \text{ for all } z \in W.$$

Hence, by [1, p. 202, Theorem 6], a variant of Hartogs' theorem holds in the Nash case. Namely, f is a Nash function if and only if it is a separately Nash function. Obviously, *separately Nash* is a counterpart of separately holomorphic. The cited result of [1] can be transferred to functions on open subsets of algebraic manifolds.

Proposition 2.4. *Let X be a complex algebraic manifold and let $f : U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$ the restriction $f|_{U \cap C}$ is a Nash function. Then f is a Nash function.*

Proof. With notation as in the proof of Proposition 2.3, the holomorphic chart

$$\varphi_p : U_p \to W_p \subset \mathbb{C}^n$$

is actually a Nash isomorphism. Hence, the composite function $f \circ \varphi_p^{-1} : W_p \to \mathbb{C}$ is a Nash function since it is a separately Nash function. It follows that f is a Nash function, as asserted. □

Proof of Theorem 1.2. We can repeat the proof of Theorem 1.1, substituting Proposition 2.4 for Proposition 2.3. □

It remains to consider regular functions. We will make use of Bertini's theorem [8] to produce connected, nonsingular algebraic curves. Given integers k and N, with $1 \leq k \leq N$, we let $\mathbb{G}_k(\mathbb{P}^N)$ denote the Grassmann manifold of k-dimensional projective subspaces of $\mathbb{P}^N$.

Let X be a complex algebraic manifold of dimension $d \geq 1$, and let W_1, W_2 be nonempty open subsets of X. Assume that the manifold X is connected (which in this case is the same as irreducible). Then there exists a connected, nonsingular algebraic curve $C \subset X$ such that $C \cap W_i \neq \emptyset$ for $i = 1, 2$. This assertion is obvious if $d = 1$, so suppose that $d \geq 2$. We may assume that X is a Zariski locally closed subset of $\mathbb{P}^N$ for some N. Setting $k = N - d + 1$, by Bertini's theorem, the intersection $C = X \cap \Lambda$ of X with a suitable projective subspace $\Lambda \in \mathbb{G}_k(\mathbb{P}^N)$ is an algebraic curve with the required properties.

Lemma 2.5. *Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic manifolds and let $f : U \to \mathbb{C}$ be a function defined on an open subset U of X. Let $\{U_\alpha\}_{\alpha \in A}$ be the collection of all connected components of U. Assume that the following two conditions hold:*

(a) *The restriction $f|_{U_\alpha}$ is a regular function for all $\alpha \in A$.*
(b) *For each nonsingular algebraic curve $C \subset X$, parallel to one of the factors of X, the restriction $f|_{U \cap C}$ is a regular function.*

Then f is a regular function.

Proof. It suffices to consider the case in which all the X_i are connected, with $\dim X_1 \geq 1$. By assumption (a), for each $\alpha \in A$ there exists a rational function R_α on X such that

$$U_\alpha \subset X^\alpha := X \setminus \operatorname{Pole}(R_\alpha) \quad \text{and} \quad f|_{U_\alpha} = R_\alpha|_{U_\alpha}.$$

Clearly, if $R_\alpha = R_\beta$ for all $\alpha, \beta \in A$, then f is a regular function, and the proof is complete.

Suppose to the contrary that $R_\mu \neq R_\nu$ for some $\mu, \nu \in A$. It follows that $R_\mu(p) \neq R_\nu(p)$ for some point $p = (p_1, \ldots, p_n)$ in $U_\mu \cap X^\nu$, and hence the set

$$W = \{x \in U_\mu \cap X^\nu : R_\mu(x) \neq R_\nu(x)\}$$

is an open neighborhood of p in X.

Observe that there exists a connected, nonsingular algebraic curve $C \subset X$, parallel to the 1^{st} factor of X, such that

$$W \cap C \neq \emptyset \quad \text{and} \quad U_\nu \cap C \neq \emptyset.$$

Indeed, we can obtain such a curve C of the form $C = C_1 \times \{p_2\} \times \cdots \times \{p_n\}$, where $C_1 \subset X_1$ is a suitably chosen connected, nonsingular algebraic curve, having nonempty intersections with the images of W and U_ν under the canonical projection from X onto X_1 (see the discussion in the paragraph preceding Lemma 2.5). By assumption (b), there exists a rational function R_C on C such that

$$U \cap C \subset C^0 := C \setminus \operatorname{Pole}(R_C) \quad \text{and} \quad f|_{U \cap C} = R_C|_{U \cap C}.$$

Note that both $U_\mu \cap C^0$ and $U_\nu \cap C^0$ are nonempty open subsets of C^0. Since

$$R_\mu|_{U_\mu \cap C^0} = f|_{U_\mu \cap C^0} = R_C|_{U_\mu \cap C^0},$$

we get

$$R_\mu|_{X^\mu \cap C^0} = R_C|_{X^\mu \cap C^0}.$$

Similarly, we have

$$R_\nu|_{X^\nu \cap C^0} = R_C|_{X^\nu \cap C^0}.$$

Consequently,

$$R_\mu|_{X^\mu \cap X^\nu \cap C^0} = R_\nu|_{X^\mu \cap X^\nu \cap C^0}.$$

The last equality leads to a contradiction since $W \cap X^\mu \cap X^\nu \cap C^0 \neq \emptyset$. $\square$

Let X be a complex algebraic manifold and let $f\colon U \to \mathbb{C}$ be a function defined on an open subset U of X. We say that a rational function R on X is a *rational representation* of f if there exists a Zariski open dense subset $X^0 \subset X$ such that

$$X^0 \subset X \setminus \mathrm{Pole}(R) \quad \text{and} \quad f|_{U \cap X^0} = R|_{U \cap X^0}.$$

Since X is nonsingular, any continuous function on U that admits a rational representation is actually regular by the Riemann extension theorem.

Proposition 2.6. *Let X be a complex algebraic manifold and let $f\colon U \to \mathbb{C}$ be a function defined on an open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$ the restriction $f|_{U \cap C}$ is a regular function. Then f is a regular function.*

Proof. We may assume that the manifold X is connected and $d = \dim X \geq 2$. In view of Lemma 2.5 (with $n = 1$), it suffices to consider the case in which the set U is connected. By Proposition 2.4, f is a Nash function. Hence, since U is connected, the graph of f is contained in an irreducible algebraic hypersurface $Y \subset X \times \mathbb{C}$. The function f admits a rational representation if and only if $\pi\colon Y \to X$, the restriction of the canonical projection $X \times \mathbb{C} \to X$, is a birational map. Suppose that π is not birational, that is, it has degree $m > 1$. We obtain a contradiction as follows. We may assume that X is a Zariski locally closed subset of $\mathbb{P}^N$ for some N. Set $k = N - d + 1$. By Bertini's theorem, for a general projective subspace $\Lambda \in \mathbb{G}_k(\mathbb{P}^N)$ both $X \cap \Lambda$ and $\pi^{-1}(X \cap \Lambda)$ are irreducible algebraic curves. Clearly, the restriction $\pi_\Lambda\colon \pi^{-1}(X \cap \Lambda) \to X \cap \Lambda$ of π has degree m. We can choose Λ so that the curve $C = X \cap \Lambda$ is nonsingular and $U \cap C \neq \emptyset$. The function $f|_{U \cap C}$ does not admit a rational representation since its graph is contained in $\pi^{-1}(C)$. However, by assumption, $f|_{U \cap C}$ is a regular function, so we get a contradiction. Thus, f is a Nash function which admits a rational representation. In conclusion, f is a regular function. $\square$

Proof of Theorem 1.3. In view of Lemma 2.5, we may assume that the set U is connected. Pick a point $p = (p_1, \ldots, p_n)$ in $X = X_1 \times \cdots \times X_n$. For $i = 1, \ldots, n$, let

$$X(i) = Y_{i1} \times \cdots \times Y_{in},$$

where $Y_{ii} = X_i$ and $Y_{ij} = \{p_j\}$ for $i \neq j$. According to Proposition 2.6, the restriction $f|_{U \cap X(i)}$ is a regular function. Since p is an arbitrary point of X, the function f is regular by [7, Theorem 7.3]. $\square$

We conclude this section by presenting two improvements upon some results of [7]. The following is a sharpening of [7, Theorem 7.2].

Proposition 2.7. *Let X be a complex algebraic manifold and let $f\colon U \to \mathbb{C}$ be a function defined on a connected open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$ and each point $x \in U \cap C$ there exists an open neighborhood $U_x \subset U$ of x such that the restriction $f|_{U_x \cap C}$ is a regular function. Then f is a regular function.*

Proof. Clearly, the restriction $f|_{U_x \cap C}$ is a Nash function for every point $x \in U \cap C$, which means that $f|_{U \cap C}$ is a Nash function. Hence, by Proposition 2.4, f is a Nash function. Now we can argue as in the proof of Proposition 2.6. □

Finally, [7, Theorem 7.3] can be sharpened as follows.

Theorem 2.8. *Let $X = X_1 \times \cdots \times X_n$ be the product of n complex algebraic manifolds and let $f\colon U \to \mathbb{C}$ be a function defined on a connected open subset U of X. Assume that for each nonsingular algebraic curve $C \subset X$, parallel to one of the factors of X, and each point $x \in U \cap C$ there exists an open neighborhood $U_x \subset U$ of x such that the restriction $f|_{U_x \cap C}$ is a regular function. Then f is a regular function.*

Proof. We can argue as in the proof of Theorem 1.3, substituting Proposition 2.7 for Proposition 2.6. □

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